# Supplementary material

Paul Plucinsky[a], Benjamin A. Kowalski[b], Timothy J. White[b], and Kaushik Bhattacharya[c]

[a]Aerospace and Engineering Mechanics, University of Minnesota
[b]Materials and Manufacturing Directorate, Air Force Research Lab
[c]Engineering and Applied Sciences, California Institute of Technology

March 6, 2018

## Supplementary videos

The video nobias.mov shows the degeneracy of a pyramid designed with no bias (i.e., no twists near the ridges). The pyramid can easily be manipulated to be stable in either the up or the down states.

The video bias.mov shows the robustness of the pyramid designed with a bias that is shown in Figure 5. The video shows that the pyramid promptly returns to its programmed direction even after being manipulated by hand into the "wrong" configuration.

# Supplementary material on nonisometric origami

## Formulation of compatibility for general building blocks

Generic nonisometric origami designs are made up of building blocks where straight interfaces, which separate regions of distinct constant director, merge to a point as sketched in Figure 1. (These interfaces, if designed appropriately, become the ridges of the origami structure upon actuation.) Each of these building blocks must independently satisfy the metric constraint for the whole origami structure to be compatible upon actuation.

To formulate this notion of compatibility of building blocks, we consider an initially flat $k$-faced building block $\omega_{\tilde{\mathbf{p}}} = \cup_{\alpha=1,\dots,k} S_\alpha \subset \mathbb{R}^2$ in which $k$ sectors $S_\alpha$ of piecewise-constant director merge to a point $\tilde{\mathbf{p}} \in \mathbb{R}^2$, which is in the interior of $\omega_{\tilde{\mathbf{p}}}$. We let $\tilde{\mathbf{t}}_\alpha \in \mathbb{S}^1$ denote the outward tangent vector at the point $\tilde{\mathbf{p}}$ describing the interface between the sectors $S_\alpha$ and $S_{\alpha+1}$ of director $\mathbf{n}_{0\alpha}$ and $\mathbf{n}_{0(\alpha+1)}$, respectively (where we define $\mathbf{n}_{0(k+1)} := \mathbf{n}_{01}$ and likewise for $S_{k+1}$). The notation is also provided in Figure 1.

Given all this, there exists a design and deformation at the building block $\omega_{\tilde{\mathbf{p}}}$ which satisfies the metric constraint if and only if the collection of interfaces $\{\tilde{\mathbf{t}}_1, \dots, \tilde{\mathbf{t}}_k\}$ and directors $\{\mathbf{n}_{01}, \dots, \mathbf{n}_{0k}\}$ satisfies

$$\begin{gathered} \mathbf{R}_\alpha (\boldsymbol{\ell}^{1/2}_{\mathbf{n}_{0\alpha}})_{3\times 2} \tilde{\mathbf{t}}_\alpha = \mathbf{R}_{\alpha+1} (\boldsymbol{\ell}^{1/2}_{\mathbf{n}_{0(\alpha+1)}})_{3\times 2} \tilde{\mathbf{t}}_\alpha, \\ \alpha \in \{1, \dots, k\} \end{gathered} \tag{1}$$

for some $\{\mathbf{R}_1, \dots, \mathbf{R}_k\} \in SO(3)$, where $(\boldsymbol{\ell}^{1/2}_{\mathbf{n}_{0\alpha}})_{3\times 2} := r^{-1/6}(\mathbf{I}_{3\times 2} + (r^{1/2} - 1)\mathbf{n}_{0\alpha} \otimes \tilde{\mathbf{n}}_{0\alpha})$ for each $\alpha \in \{1, \dots, k\}$. (Again, we define $\mathbf{R}_{k+1} := \mathbf{R}_1$.) We have addressed the intimate connection between this compatibility and the metric constraint (1) elsewhere [1, 2], as well as the justification of these configurations as designable actuation. Thus here, we focus on examples which highlight the richness of the design landscape for building blocks.

In this direction, we note that if the equations (1) are solved for some collection $\{\mathbf{R}_\alpha\} \in SO(3)$, then any deformation $\mathbf{y} \colon \omega_{\tilde{\mathbf{p}}} \to \mathbb{R}^3$ of the form

$$\begin{gathered} \mathbf{y}(\tilde{\mathbf{x}}) = \mathbf{Q}\mathbf{R}_\alpha (\boldsymbol{\ell}^{1/2}_{\mathbf{n}_{0\alpha}})_{3\times 2} (\tilde{\mathbf{x}} - \tilde{\mathbf{p}}) + \mathbf{y}(\tilde{\mathbf{p}}), \quad \tilde{\mathbf{x}} \in S_\alpha, \\ \alpha \in \{1, \dots, k\}, \quad \mathbf{Q} \in SO(3) \quad \text{and} \quad \mathbf{y}(\tilde{\mathbf{p}}) \in \mathbb{R}^3 \end{gathered} \tag{2}$$

realizes the metric on $\omega_{\tilde{\mathbf{p}}}$ and is, consequently, a candidate for the deformation of the building block upon actuation. Furthermore, we can use the freedom afforded us in the rigid rotation $\mathbf{Q}$ and translation $\mathbf{y}(\tilde{\mathbf{p}})$ of the building block to attempt to link multiple such building blocks together to actuate complex shapes. This framework permits a systematic investigation: (i) characterize the $\{\mathbf{t}_\alpha\}$, $\{\mathbf{n}_{0\alpha}\}$ and $\{\mathbf{R}_\alpha\}$ which satisfy (1), (ii) use this characterization to build nonisometric origami building blocks described by deformations of the form in (2), and (iii) use these building blocks as building blocks to form more complex shapes.

## Some efforts on characterization

For the characterization herein, we focus on the case that each of the directors $\mathbf{n}_{0\alpha} \in \mathbb{S}^2$ is planar (i.e., $\mathbf{n}_{0\alpha} \cdot \mathbf{e}_3 = 0$ for each $\alpha$). This encapsulates all such nonisometric origami building blocks which can be realized using current experimental techniques (e.g., the voxelation technique of Ware et al. [4]). We also assume that $\mathbf{n}_{0\alpha}$ and $\mathbf{n}_{0(\alpha+1)}$ are linearly independent (otherwise, the interface $\tilde{\mathbf{t}}_\alpha$ would be superfluous). Finally, we assume that the sheet is under actuation (i.e., $r \neq 1$).

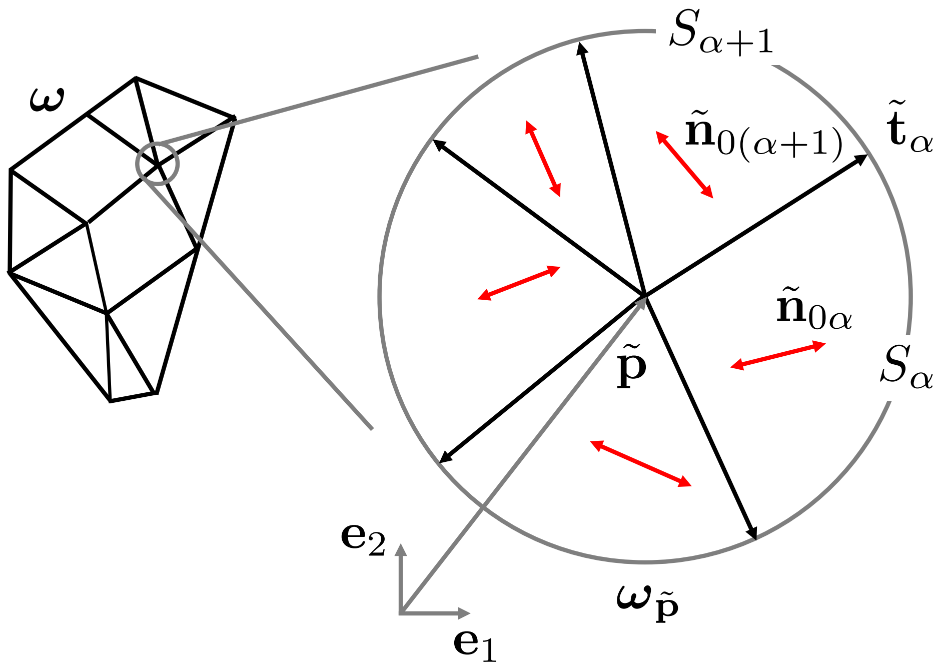


Figure 1: At each vertex, the actuation induced shape change of the sectors surrounding the vertex must be comapatible for the pattern to be nonisometric origami.

Under these assumptions, we can satisfy (1) only if the set of tangent vectors $\{\tilde{\mathbf{t}}_\alpha\} \in \mathbb{S}^1$ satisfies

$$\begin{aligned} &\tilde{\mathbf{t}}_\alpha \in \operatorname{span}\{\tilde{\mathbf{n}}_{0\alpha} + \tilde{\mathbf{n}}_{0(\alpha+1)}, \tilde{\mathbf{n}}_{0\alpha} - \tilde{\mathbf{n}}_{0(\alpha+1)}\}, \\ &\qquad \alpha \in \{1, \ldots, k\}. \end{aligned} \tag{3}$$

In words, compatibility requires that the tangent vector at each interface bisect the two corresponding planar directors (up to a reflection of one or both of the directors). This follows from taking the squared norm of both sides of (1), and manipulating this quantity using the stated assumptions.

This condition on the interface tangents is necessary for a compatible nonisometric origami building block, but it is not sufficient. For sufficiency, we need (additionally) to find a set of rotations $\{\mathbf{R}_\alpha\} \in SO(3)$ which satisfies (1). In this direction, we note that all solutions to three-faced building blocks of nonisometric origami have been worked out explicitly in the thesis of Plucinsky [3]. For instance, if we are given any three distinct planar director $\{\tilde{\mathbf{n}}_{01}, \tilde{\mathbf{n}}_{02}, \tilde{\mathbf{n}}_{03}\} \in \mathbb{S}^1$, then there are generically 16 nonisometric origami building block designs associated to these directors that are compatible for heating. These design are bi-stable, (i.e., the pattern can actuate either up or design). The formulas associated to the designs and actuations are cumbersome, but explicitly stated in Appendix A of the thesis and easily amenable to calculation (in mathematica, for instance). We showcase the rich design landscape for three faced building blocks by considering one such example below.

Now in trying to generalize these results to ($k > 3$)-faced building blocks, there are important distinctions to be made. Most notably, such building blocks, when compatible, are generically and non-trivially degenerate. Specifically, if we can find a ($k > 3$)-faced building block which has a solution to (1) upon actuation, then there are, in fact, continuous families of solutions. The basic idea is that additional interfaces, beyond three, add degrees of freedom to the systems (i.e., we pick up an extra folding angle rotation for every additional interface). These extra degrees of freedom allow for continuous families of solutions. Indeed, recall that for the symmetric four-faced building block design, we showed that the interfaces can be continuously deformed from the (initially symmetric) pyramid to generate an entire family of "metric invariant" actuations arising from this design (Figure 3(c) in the main text). We emphasize again that this is a generic fact of ($k > 3$)-faced building blocks (though we will refrain from supplying the formal argument here).

From a design point of view, these result have important implications: due to the degeneracies of the nonisometric building blocks, we need a strategy that breaks the up-down symmetry and distinguishes between various metric-invariant actuations associated to building blocks with more that three interfaces. We believe that in tailoring the twist angles $\tau$ and the width of the wedges $w$, our strategy to bias the interfaces can enable this simultaneous functionality.

## Examples: General three-faced building blocks

To highlight the richness of the design landscape for nonisometric origami building blocks, we consider a single set of directors,

$$\{\mathbf{n}_{01}, \mathbf{n}_{02}, \mathbf{n}_{03}\} = \{\mathbf{e}_1, \ \cos(\frac{5\pi}{36})\mathbf{e}_1 + \sin(\frac{5\pi}{36})\mathbf{e}_2, \ \cos(\frac{5\pi}{18})\mathbf{e}_1 - \sin(\frac{5\pi}{18})\mathbf{e}_2\}, \tag{4}$$

and provide all the compatible nonisometric origami building blocks and corresponding actuations associated to this set. There are 16 such compatible designs, and each design and actuation is provided in Figures 2, 3, 4 and 5 (explicit formulas for the actuation are a direct result of Theorem A.3.6 in [3]). The actuation parameter in all the examples shown is $r = 0.5$, but the solutions exist and are continuous for all $r \in (0, 1]$. The bi-stable solutions are reflections of each other. In other words, the second solution is obtained by replacing the folding angles of the first with folding angles of the same magnitude but opposite sign. The colors are associated to a particular director: red is $\mathbf{n}_{01} = \mathbf{e}_1$, green is $\mathbf{n}_{02} = \cos(\frac{5\pi}{36})\mathbf{e}_1 + \sin(\frac{5\pi}{36})\mathbf{e}_2$, and blue is $\mathbf{n}_{03} = \cos(\frac{5\pi}{18})\mathbf{e}_1 - \sin(\frac{5\pi}{18})\mathbf{e}_2$.

We emphasize again that all these examples correspond to a *single set* of directors (notice the arrows in Figure 2(a), 3(a), 4(a) and 5(a)). If we change the set, then we would obtain a new collection of nonisometric origami building blocks, likely 16 but sometimes less if their is some symmetry in associated to the set of directors. Thus, there are an infinite number of three-faced building blocks, and this is the simplest possible case. There is still much to be explored in the direction of ($k > 3$)-faced building blocks.

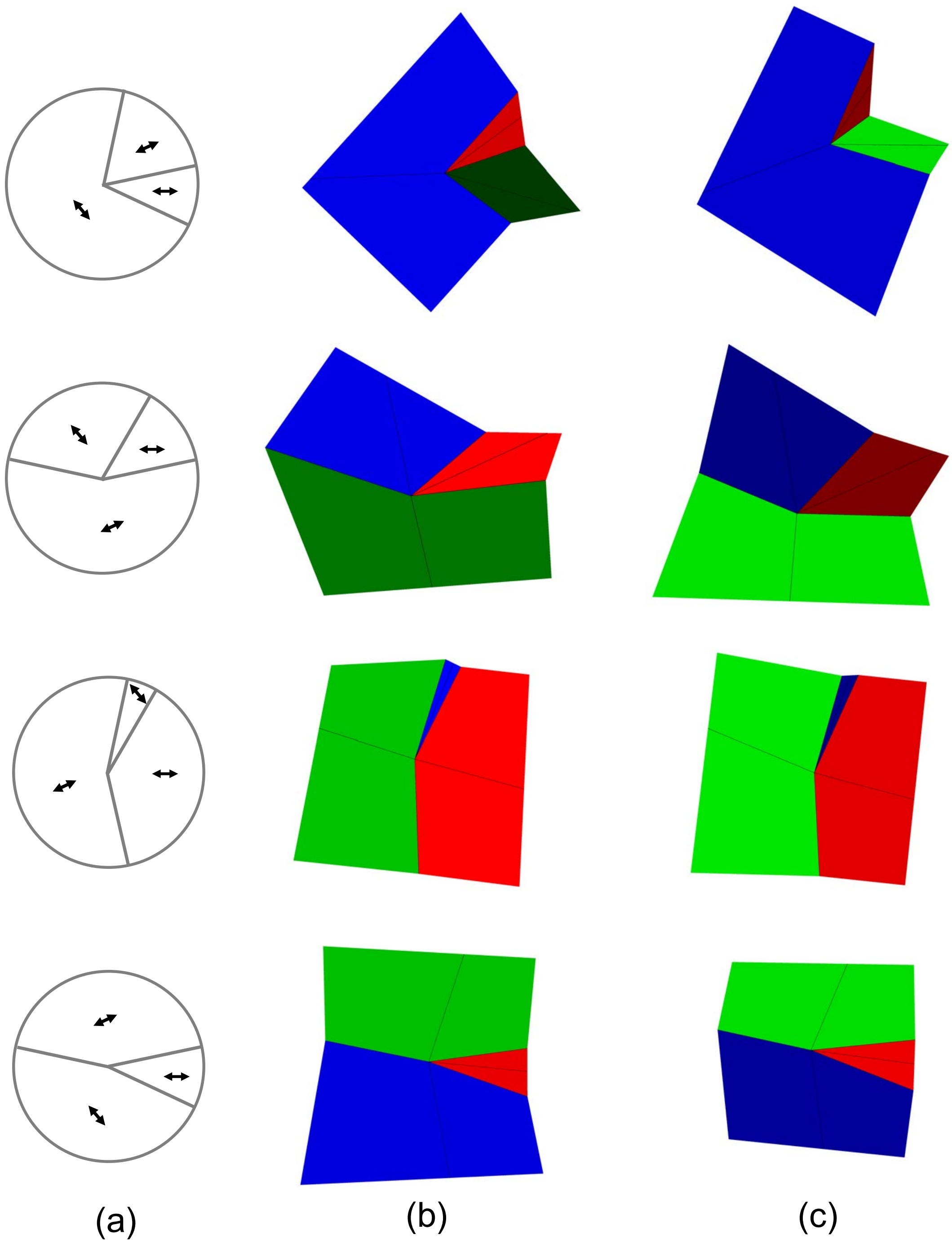


Figure 2: (a) Director design for the building block. (b),(c) The bi-stable actuations for $r = 0.5$.

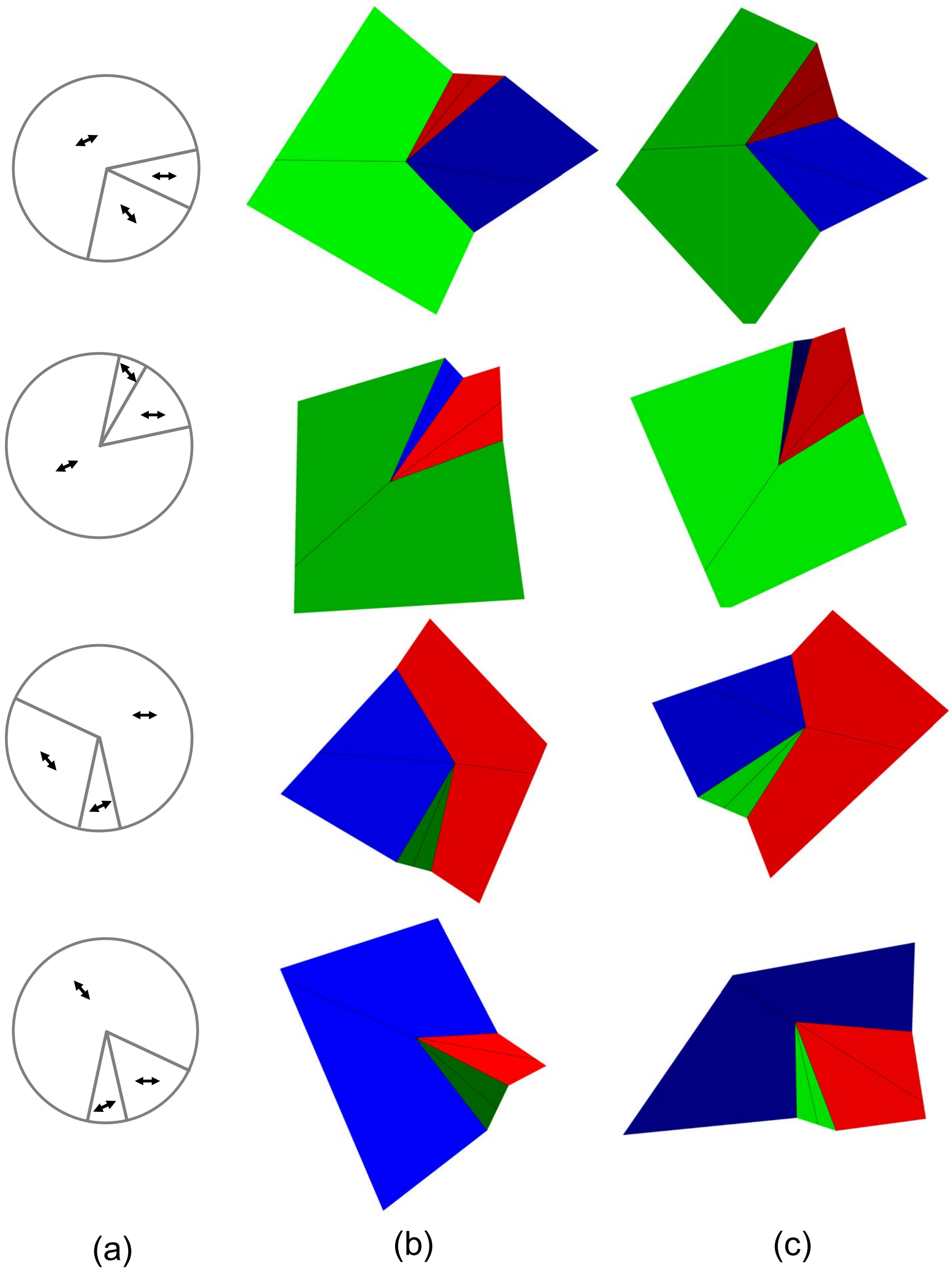


Figure 3: (a) Director design for the building block. (b),(c) The bi-stable actuations for $r = 0.5$.

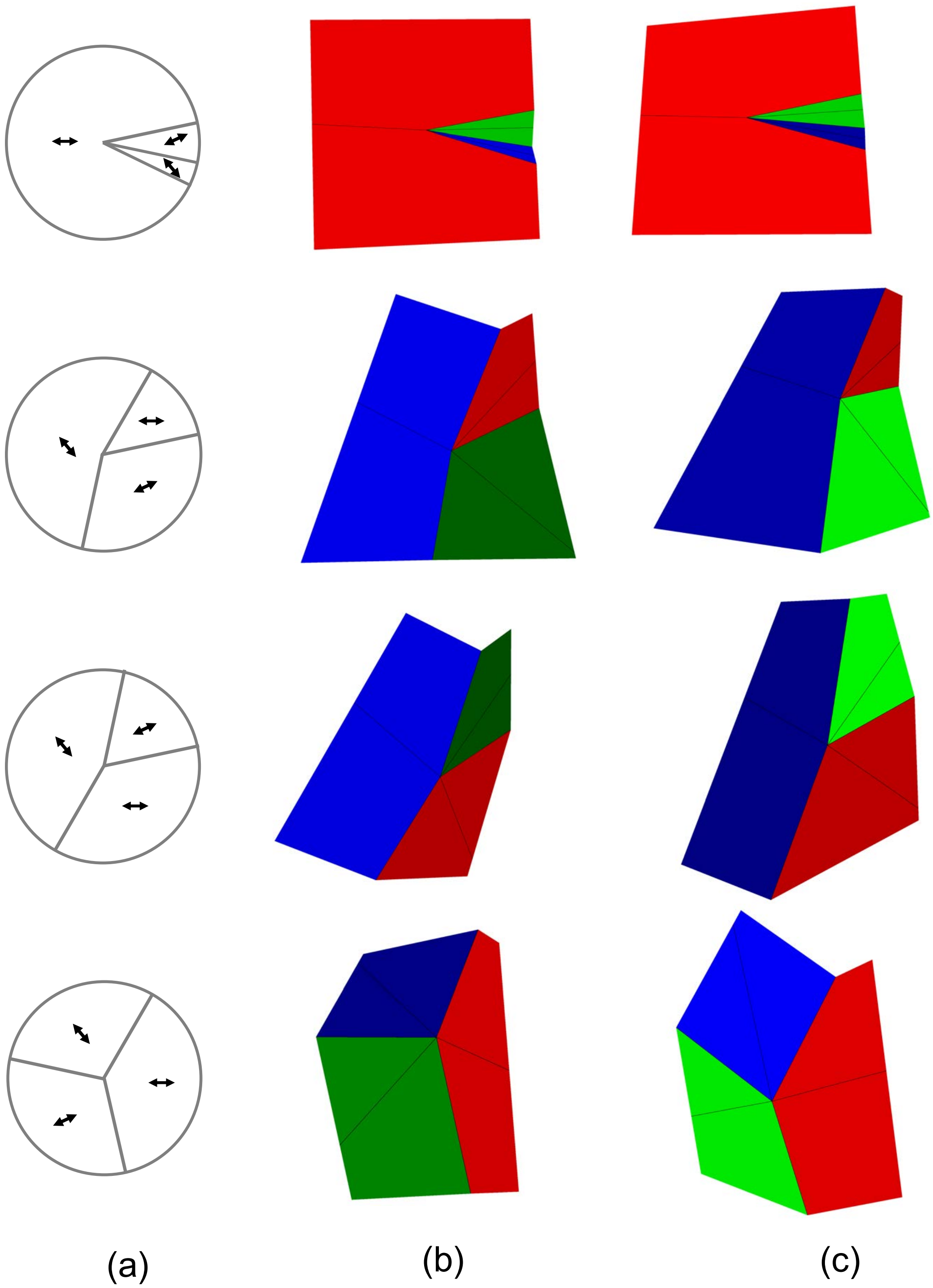


Figure 4: (a) Director design for the building block. (b),(c) The bi-stable actuations for $r = 0.5$.

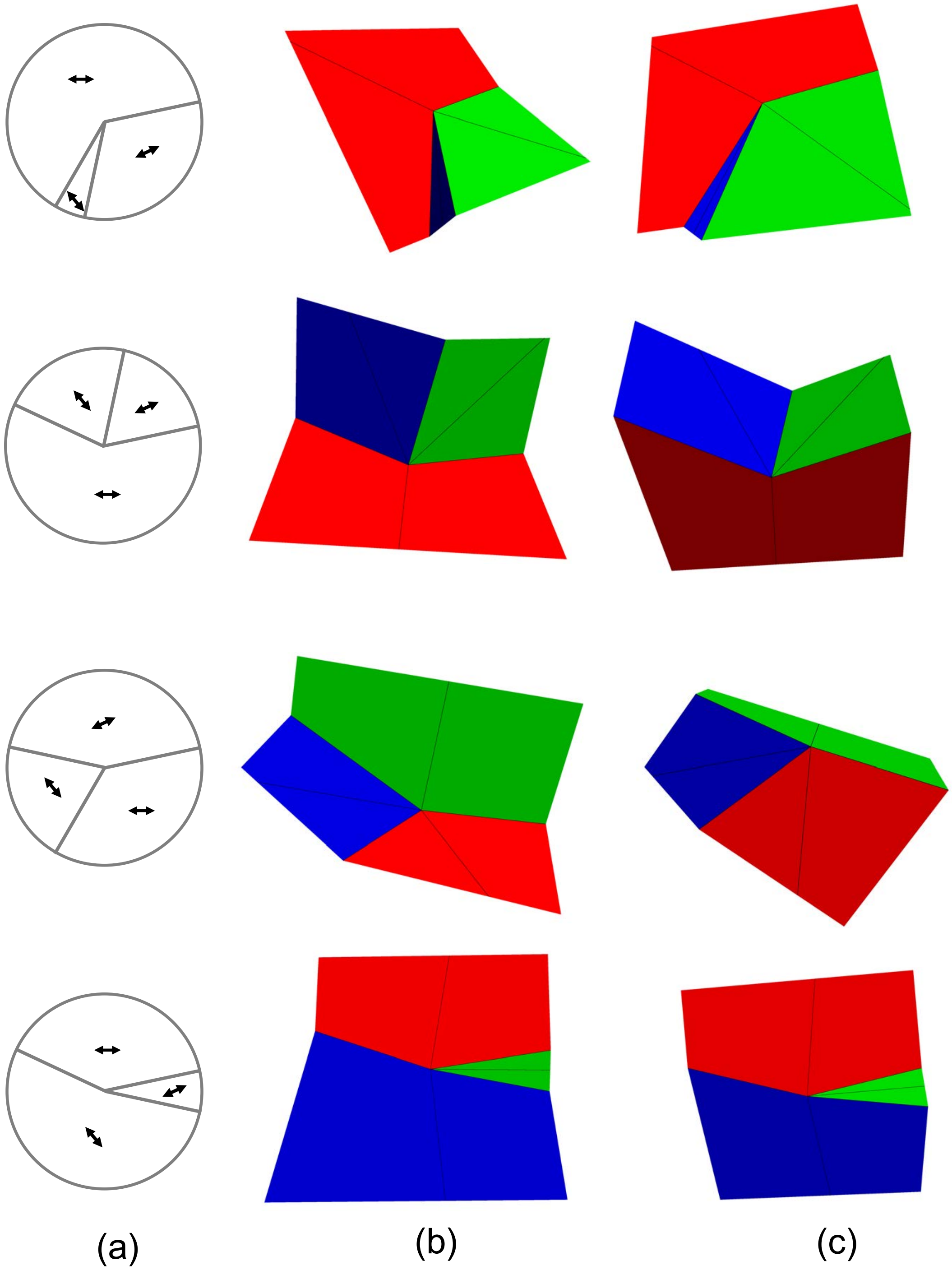


Figure 5: (a) Director design for the building block. (b),(c) The bi-stable actuations for $r = 0.5$.

## References

[1] P. Plucinsky, M. Lemm, and K. Bhattacharya. Programming complex shapes in thin nematic elastomer and glass sheets. *Physical Review E*, 94(1):010701, 2016.

[2] P. Plucinsky, M. Lemm, and K. Bhattacharya. Actuation of thin nematic elastomer sheets with controlled heterogeneity. *Archive for Rational Mechanics and Analysis*, 2017.

[3] P. P. Plucinsky. *The deformations of thin nematic elastomer sheets*. PhD thesis, California Institute of Technology, 2017.

[4] T. H. Ware, M. E. McConney, J. J. Wie, V. P. Tondiglia, and T. J. White. Voxelated liquid crystal elastomers. *Science*, 347(6225):982–984, 2015.